Section Check In – 1.01 Proof

Questions

1. Prove by exhaustion that, in the set of natural numbers less than 50, there are fewer square numbers than prime numbers.

2. Let p be a prime number such that $2 < p < 50$. Prove, by exhaustion, that for all such p, $(p-1)(p+1)$ is divisible by 8.

3. The following result is known as the "difference of two squares"

   $$\left(a^2 - b^2\right) = (a+b)(a-b).$$

   Find and prove a similar formula for the difference of two cubes $\left(a^3 - b^3\right)$.

4. Prove that $n-3$ is even if and only if n is odd.

5. A new hotel is to be built and will have a cylindrical swimming pool.
   Given that the pool's tiled surface area will be $225\,\text{m}^2$ show that its maximum volume is approximately $366\,\text{m}^3$.

Worked solutions

1. Proof by exhaustion:
All square numbers less than 50 are $1, 4, 9, 16, 25, 36, 49$
All prime numbers less than 50 are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$
Therefore there are fewer squares than primes.

2. Proof by exhaustion:

$2(4) = 8$
$4(6) = 24 = 3(8)$
$6(8)$
$10(12) = 15(8)$
$12(14) = 21(8)$
$16(18) = 36(8)$
$18(20) = 45(8)$
$22(24) = 66(8)$
$28(30) = 105(8)$
$30(32) = 120(8)$
$36(38) = 171(8)$
$40(42) = 210(8)$
$42(44) = 231(8)$
$46(48) = 276(8)$

Therefore the product of the two numbers adjacent, at either side, of the low primes (< 50) is divisible by 8.

3. $\left(a^3 - b^3\right) = (a-b)\left(a^2 + ab + b^2\right)$

Proof:

RHS $= (a-b)\left(a^2 + ab + b^2\right)$
$= a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3$
$= a^3 - b^3$
$=$ LHS

Hence proven.

4. Need $\Rightarrow$ and $\Leftarrow$ for if and only if proofs.
Therefore, initially need to prove that $n-3$ is even $\Rightarrow n$ is odd.

Now $n-3$ is even so $n-3=2a$ for $a \in \mathbb{Z}$.
So $n=2a+3=2a+2+1=2(a+1)+1$.
$2(a+1)$ is divisible by 2 and is therefore even.
Hence $2(a+1)+1=n$ is therefore odd.

Now need to prove that n odd $\Rightarrow n-3$ is even.

If n odd then $n=b+1$ for b even.
So $n-3=b+1-3=b-2$.
Now b is even $\therefore b$ is divisible by 2 and $-2=2(-1)$ is also divisible by 2.
So $n-3$ is divisible by 2 and is therefore even.

Therefore $n-3$ is even if and only if n is odd.

5. Volume is $\pi r^2 h$
Surface area is $\pi r^2 + 2\pi rh = 225$

So $h=\dfrac{225-\pi r^2}{2\pi r}$ and $V=\pi r^2\left(\dfrac{225-\pi r^2}{2\pi r}\right)=\dfrac{225}{2}r-\dfrac{\pi}{2}r^3$

For the maximum volume we need $\dfrac{\mathrm{d}V}{\mathrm{d}r}=0$, i.e. $\dfrac{\mathrm{d}V}{\mathrm{d}r}=\dfrac{225}{2}-\dfrac{3\pi}{2}r^2=0$

So $r=\sqrt{\dfrac{225}{3\pi}}$
Substituting this into our equations for h and V gives $V=366.45...\ \mathrm{m}^3$.